Optimal Redundancy Resolution of a Kinematically Redundant Manipulator for a Cyclic Task

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This article proposes a method for the global optimization of redundancy over the whole task period in a kinematically redundant manipulator. The necessary conditions based on the calculus of variations for integral-type criteria result in a second-order differential equation. For a cyclic task, the boundary conditions for conservative joint motions are discussed. Then, we reformulate a two-point boundary value problem to an initial value adjustment problem and suggest a numerical search method based on the iterative optimization for providing a globally optimal solution using the gradient projection method. Since the initial joint velocity is parameterized with the number of redundancy, we only search parameter values in the parameterized space using the configuration error between the initial and final time. We show through numerical examples that multiple nonhomotopic extremal solutions satisfying periodic boundary conditions exist according to initial joint velocities for the same initial configuration. Finally, we discuss an algorithm for topological liftings of the paths and demonstrate the generality of the proposed method by considering the dynamics of a manipulator.

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1. INTRODUCTION

A kinematically redundant manipulator possesses more degrees of freedom (dof) than necessary for performing a specified task. Redundancy by adding redundant dof to a nonredundant manipulator yields increased dexterity and versatility due to the infinite number of inverse kinematic solutions that result in the same end-effector trajectory. Therefore, the redundancy resolution of a redundant manipulator for reconfiguring the arm without affecting the end-effector position has been discussed in the framework of how to optimize some performance measures while carrying out its given task. And a number of redundancy resolution schemes for determining joint trajectories have been developed by using global or local optimization methods.

In the framework of local optimization (instantaneous optimization), Whitney\textsuperscript{1} proposed a resolved motion technique to determine the joint velocity using the Jacobian pseudoinverse. Liégeois\textsuperscript{2} suggested a modified pseudoinverse approach for avoiding joint limits. The manipulability measure is introduced by Yoshikawa,\textsuperscript{3} such as $M_m = \sqrt{\text{det}(JJ^T)}$, which may be used to avoid singular regions, where $J$ is the Jacobian matrix of the kinematic equations of a manipulator. A comprehensive review of the pseudoinverse approaches to the redundant manipulator is addressed by Klein and Huang.\textsuperscript{4} Hollerbach and Suh\textsuperscript{5} gave redundancy resolution through torque optimization. Dubey et al.\textsuperscript{6,7} discussed a resolution scheme based on the gradient projection method. Baillieul\textsuperscript{8} proposed the Extended Jacobian method to minimize or maximize an objective function by using additional constraints for optimality. Chang\textsuperscript{9} developed a closed-form solution for the inverse kinematics using the Lagrange multiplier method in which a set of equations are added as constraints on the kinematic equations. These methods based on additional constraints have conservative joint motions because of the nonredundant motion in nature. All of these local optimization methods are based on the null projection to determine the homogeneous solution of the inverse kinematic solutions. However, all of these methods have been based on the approach for utilization of redundancy through local optimization of some objective functions, so its overall effects on the performance of a manipulator are not considered.

Global optimization, on the other hand, determines a joint trajectory from a complete description of the desired end-effector trajectory, which are based on the global redundancy resolution in time with an integral-type criterion. Suh and Hollerbach\textsuperscript{10} addressed a global torque optimization method based on the calculus of variations. The globally optimal redundancy control using Pontryagin's maximum principle is considered by Nakamura and Hanafusa\textsuperscript{11} and the
optimal problem is modified to the minimal value searching problem, whose search dimension is twice larger than the degrees of redundancy when the dynamics of a manipulator is considered. Kazerounian and Wang\textsuperscript{12} also used the calculus of variations to develop global solutions for the least-squares joint velocities and the least kinetic energy. Some researchers suggest control methods based on adaptive control for control of the manipulator configuration directly in the task space without complicated inverse kinematic transformation.\textsuperscript{13,14}

Although global optimization provides a more stable solution than local optimization, it still has some undesirable effects such as nonconservative joint motions. In other words, for a given periodic task the configuration of the manipulator is not periodic after each cycle. In fact, one cannot even predict the configuration after repeated cycles. Recently, Martin et al.\textsuperscript{15} suggested a reduced-order form equivalent to a second-order differential equation in $n$ variables, where $n$ is the number of degrees of freedom of the manipulator. The reduced-order form is obtained from the necessary conditions for optimality using the null projection operator. They pointed out that the initial and final joint configurations and velocities must be the same for a cyclic task in a practical case. But, the reduced-order form is not applicable to a general performance index such as the kinetic energy. Anderson and Angeles\textsuperscript{16} handled redundancy using a constrained nonlinear least-squares minimization approach to avoid the production of nonconservative joint motions. Wampler\textsuperscript{17} developed certain inverse kinematic functions to form a cyclic tracking algorithm. To achieve the cyclic behavior of conservative solutions, a method for approximating the globally optimal solution using certain periodical functions (e.g., Fourier series) is also suggested,\textsuperscript{18} but it is only an approximate solution.

In this article, global optimization over the whole task period of a redundant manipulator for providing conservative joint motions is worked out using a general integral-type criterion. To offer a solution for global optimization, we use the calculus of variations. The necessary conditions derived by minimizing an integral-type criterion result in a second-order differential equation in $n$ variables.

To uniquely specify the optimal solution, one must consider the boundary conditions as well as the necessary conditions. For a cyclic task, the boundary conditions become periodic to produce conservative solutions. The periodic boundary conditions due to the conservativity requirement are discussed. Based on the corresponding periodic boundary problem, we refine the periodic boundary problem to an initial value adjustment problem. In the initial value adjustment problem, we only find the initial joint velocity satisfying periodic boundary conditions if the initial configuration is imposed. To find the initial joint velocity $\dot{\theta}(t_0)$, we parameterize the joint velocity with the number of redundancy. Then, we only search the parameter values using the configuration error, $\theta(t_1) - \theta(t_0)$, in the parameterized space whose dimension is the same as the number of redundancy, where $t_0$ and $t_1$ are the initial and final time, respectively. If the number of redundancy is one, we present a numerical search method based on the iterative optimization for providing the conservative opti-
mal solution. The numerical search method using the gradient projection consists of iterative adjustment of the parameter values where the convergence for the global optimality depends on the initial parameter values. We discuss a systematic assignment of an initial parameter value to determine an initial joint velocity, at which the solution satisfying the periodic boundary conditions yields path-wise optimal (i.e., globally optimal in time).

Also, we compare an optimal solution in the desirable homotopy class with an extremal solution at the same initial configuration through numerical examples. To better understand, we depict the joint trajectories of solutions on the \( \theta_3 - \theta_2 \) plane with sets of the inverse kinematic solutions according to a given task. Once an optimal solution is obtained for an imposed \( \theta(t_0) \), the parameter obtained in the optimal solution can be used in a method for topological liftings of the paths for another initial configuration while excluding extremal solutions. Finally, we show the generality of our proposed method using a three-link planar manipulator for cyclic tasks by considering the dynamics of the manipulator. The motions of the manipulator in the case of the least kinetic energy are depicted to compare with those of the least-squares joint velocities.

The article is organized as follows: In Section 2, the necessary conditions are derived by using the Euler–Lagrange equations and periodic boundary conditions are discussed. A numerical search method that satisfies conservativity requirement for a cyclic task, as well as the necessary conditions, is suggested in Section 3. In Section 4, we discuss some features in optimal redundancy resolution through numerical examples. In these examples, we address a general algorithm for obtaining a globally optimal solution. And, a method of topological liftings of the paths for a given task is addressed. Section V discusses the results of this article and draws some conclusions.

2. GLOBAL OPTIMIZATION OF REDUNDANCY

2.1. Necessary Conditions for Optimality

For a global optimization of redundancy, one may consider an integral-type performance criterion subject to the kinematic constraints. The problem is set up as follows:

\[
\text{minimize } r = \int_{t_0}^{t_1} p(\theta, \dot{\theta}, t) \, dt
\]

subject to the kinematic constraints

\[
x(t) = f(\theta(t)),
\]

where \( \theta(t) \in \mathbb{R}^n \) is the joint vector and \( x(t) \in \mathbb{R}^m \) represents the position and orientation of the end effector. This is a functional optimization problem with equality constraints and may be solved by applying the calculus of variations. The problem is to find a joint trajectory \( \theta(t) \), \( t_0 \leq t \leq t_1 \), that minimizes the
performance index (1) among joint trajectories realizing the desired end-effector trajectory of the task (2). To obtain the necessary conditions for optimality with equality constraints, we introduce Lagrange multipliers $\lambda \in \mathbb{R}^m$ and use the Lagrangian function. The necessary conditions for minimizing the general integral-type criterion are derived from the variation of the augmented objective function using Lagrange multipliers. For convenience, throughout this article the argument $t$ is sometimes omitted when no confusion is likely to arise.

The augmented objective function using the Lagrangian function is defined as

$$r^* = \int_{t_0}^{t_f} L(\theta, \dot{\theta}, \lambda, t) \, dt$$

where $L(\theta, \dot{\theta}, \lambda, t) = p(\theta, \dot{\theta}, t) + \lambda^T[x - f(\theta)]$ is the Lagrangian function. The variation of the augmented objective function for specified final time is

$$\delta r^* = \int_{t_0}^{t_f} \left[ \frac{\partial L}{\partial \theta} \frac{d}{dt} - \frac{\partial L}{\partial \dot{\theta}} \right] \delta \theta \, dt + \left[ \frac{\partial L}{\partial \theta(t_f)} \right]^{R} \delta \theta(t_f) - \left[ \frac{\partial L}{\partial \theta(t_0)} \right]^{T} \delta \theta(t_0) = 0$$

The above entails the necessary conditions with boundary conditions that the joint trajectory should satisfy. The first term of the right side of eq. (4) is the necessary condition for optimality of (1) and (2), i.e., the matrix form of the Euler–Lagrange equations, and for arbitrary $\delta \theta$ we have

$$\frac{\partial L}{\partial \theta} \frac{d}{dt} - \frac{\partial L}{\partial \dot{\theta}} = 0$$

(5)

For a reasonable candidate for $p(\theta, \dot{\theta}, t)$, we choose the following function.

$$p(\theta, \dot{\theta}, t) = \frac{1}{2} \dot{\theta}^T M(\theta) \dot{\theta} + \rho g(\theta)$$

(6)

where $\rho$ is a scalar value and $g(\theta)$ is some function of configuration such as the manipulability measure\(^3\) or a potential function that gives large value in the neighborhood of obstacles. If $M \in \mathbb{R}^{n \times n}$ is a configuration-independent diagonal matrix, the problem becomes the minimization of the weighted norm of joint velocities. If one wishes to minimize kinetic energy, $M(\theta)$ must be an inertial matrix of the dynamic equation of a manipulator. In this case, the optimization problem becomes the minimization of kinetic energy.\(^12,14\)

The necessary condition (5) for the performance index (6) yields

$$M \ddot{\theta} + M \dot{\theta} - \rho \dot{\theta} = J^T \lambda - \frac{\partial}{\partial \theta} \left( \frac{1}{2} \dot{\theta}^T M \ddot{\theta} \right) = 0$$

(7)

where $J \in \mathbb{R}^{m \times n}$ is the Jacobian matrix of (2) and $g_\theta$ is the gradient vector of $g(\theta)$. Since the optimal redundancy resolution is achieved at the acceleration
level, we differentiate the kinematic constraints (2) twice; then, it results in the acceleration constraints
\[ \ddot{\mathbf{x}} - J \ddot{\mathbf{\dot{\theta}}} - J \dot{\mathbf{\dot{\theta}}} = 0 \] (8)

Using eqs. (7) and (8), the trajectory that satisfies the Euler–Lagrange equations is obtained by eliminating Lagrange multipliers as
\[ \ddot{\mathbf{\dot{\theta}}} = J^T_M(\mathbf{x} - J \dot{\mathbf{\dot{\theta}}}) - (I - J^T_M J)M^{-1}\left[ \dot{\mathbf{\dot{\mathbf{\dot{\theta}}}}} - \frac{\partial}{\partial \theta} \left( \frac{1}{2} \dot{\mathbf{\dot{\theta}}}^T M \dot{\mathbf{\dot{\theta}}} \right) - \rho g \right] \] (9)

where \( I \in \mathbb{R}^{n \times n} \) is an identity matrix and \( J^T_M = M^{-1}J^T(JM^{-1}J^T)^{-1} \) is a weighted pseudoinverse of \( J \). Equation (9) is a second-order differential equation that should be satisfied to minimize the performance index (6) regardless of the boundary conditions.

In the Lagrangian formulation of manipulator dynamics, \( \dot{\mathbf{\dot{\theta}}} - \partial / \partial \theta (1/2 \dot{\mathbf{\dot{\theta}}}^T M \dot{\mathbf{\dot{\theta}}}) \) yields \( \mathbf{V}_{cc} \), where \( \mathbf{V}_{cc} \) is the vector of Coriolis and centrifugal torques. Therefore, if \( M(\theta) \) in eq. (6) is the inertial matrix, (9) may be represented as
\[ \ddot{\mathbf{\dot{\theta}}} = J^T_M(\mathbf{x} - J \dot{\mathbf{\dot{\theta}}}) - (I - J^T_M J)M^{-1}(\mathbf{V}_{cc} - \rho g \dot{\theta}) \] (10)

Under the assumption that \( J \) has full rank, \( J^T_M \) exists. If the rank deficiency of \( J \) occurs, then it is a singular configuration. To avoid singular regions in a path, we may choose \( g(\theta) \) as the manipulability measure, then we would generate singular-free optimal trajectories as \( p \) increases. Also, we note that the term \( (I - J^T_M J) \) in (10) projects \( M^{-1}(\mathbf{V}_{cc} - \rho g \dot{\theta}) \) onto the null space of the Jacobian matrix. Therefore, any motion in the null space cause no motion of the end effector.

2.2. Boundary Conditions

To uniquely specify the optimal solution of a second-order differential equation (10) in \( n \) variables, the boundary conditions for \( 2n \) variables, as well as the necessary conditions, are required. The boundary conditions of the optimal redundancy resolution for a redundant manipulator have been discussed extensively by several researchers. Among such boundary conditions, two cases are important in a practical sense: natural boundary conditions and periodic boundary conditions. Since the kinematic constraints should be satisfied to achieve a task, the self-evident boundary conditions are
\[ \mathbf{x}(t_b) = \mathbf{f}(\theta(t_b)) \quad t_b = t_0 \text{ or } t_1 \] (11)

Natural boundary problem is no requirement placed on the initial or final joint configurations aside from the kinematic constraints (e.g., free endpoints). Therefore, \( \delta \theta(t_b) \) in (4) is no longer vanished. To satisfy (4), the following condition should be satisfied
From (12), we can conclude that $\delta \theta(t_b)$ is normal to a linear combination of the gradient vectors of the constraints (11). This means that the natural boundary conditions become

$$N_M^T \dot{\theta}(t_b) = 0$$

(13)

where $N_M = MN \in \mathbb{R}^{n \times n-m}$ is a weighted null space matrix and $N \in \mathbb{R}^{n \times n-m}$ is any matrix whose columns span the null space of $J$. Therefore, the natural boundary conditions yield

$$\dot{\theta}(t_b) = J_M^T \dot{x}(t_b)$$

(14)

The importance of the solution (10) subject to the natural boundary conditions (14) lies in the fact that the solution will give the minimum value of the performance index (1) compared with the solution for the specified boundary conditions. If the workspace trajectory, however, is periodic, i.e., $x(t_1) = x(t)$, then the joint trajectory obtained by solving the natural boundary conditions is not cyclic. This problem is generally known as the nonconservativity of the solution. To satisfy the conservativity requirement for a cyclic task, we must seek to joint trajectories to minimize (1) subject to the forced boundary conditions $\theta(t_1) = \theta(t_0)$ and $\dot{\theta}(t_1) = \dot{\theta}(t_0)$. And, if $\theta(t_1) = \theta(t_0)$ is satisfied for arbitrary $\theta(t)$, then its variation at both endpoints $t_0$ and $t_1$ must be equal. This condition can be obtained to the form of $\dot{\theta}(t_0) = \dot{\theta}(t_1)$ from (4). These two constraints become the periodic boundary conditions. Therefore, eq. (10) subject to $\theta(t_1) = \theta(t_0)$ and $\dot{\theta}(t_0) = \dot{\theta}(t_1)$ yields the globally optimal solution satisfying the periodic boundary conditions for a cyclic task. Even though the optimization problem with $2n$ unspecified values of $\theta(t_0)$ and $\dot{\theta}(t_0)$ is interesting, it is not easy to obtain a solution according to periodic boundary conditions for conservative joint motions as pointed out by Wang and Kazeronian. Since the boundary value problem can be treated according to different preassigned kinematic requirements, we consider only the imposed initial joint configuration in this article. Then, the joint values at both boundaries are imposed to satisfy periodic boundary conditions. We reformulate this optimization problem as an initial value adjustment problem to find an initial joint velocity for the imposed initial joint value.

3. NUMERICAL SEARCH METHOD TO DETERMINE INITIAL JOINT VELOCITY

In general, the end-effector velocity can be related to the joint velocity by the well-known Jacobian relation $\dot{x} = J \dot{\theta}$. If $J$ is a square matrix ($n = m$) and has a full column rank, the joint velocity $\dot{\theta}$ required to achieve the desired end-effector motion can be directly solved. However, for redundant manipulators
(\(n > m\)), the general solution of the Jacobian relation may be obtained by using the pseudoinverse of the Jacobian matrix as follows

\[
\dot{\theta} = J^+ \dot{x} - \alpha (I - J^+J) \nabla H
\]  

(15)

where \(\alpha\) is a scalar value and \(\nabla H\) is the gradient vector of an arbitrary objective function \(H\). This method is called the gradient projection method. Some approaches for the proper selection of \(H\) for singularity avoidance or obstacle avoidance have been proposed.\(^2\) The joint velocity is composed of two parts: The first term of the right side in (15) is the particular solution due to the task and the second part is the homogeneous solution due to the objective function, which belongs to the null space of the Jacobian matrix or the homogeneous solution space.

Since our optimization problem is refined to an initial value adjustment problem for finding an initial joint velocity \(\dot{\theta}(t_0)\), we only find a homogeneous solution, at which we can obtain a conservative joint trajectory. Indeed, if we manipulate \(n\) variables at the initial endpoint based on the initial value adjusting method, we have to repeat the estimation and modification of \(n\) values, which requires a lot of computation. However, if \(\dot{\theta}(t_0)\) in (15) satisfies \(\dot{x}(t_0) = J\dot{\theta}(t_0)\), then \(\dot{\theta}(t)\) necessarily satisfies \(\dot{x}(t_i) = J\dot{\theta}(t_i)\) as long as \(\dot{\theta}(t)\) is governed by (10).

To parameterize the initial joint velocity, we use a null basis vector. If the redundancy is more than one, we must carry out \(n - m\) dimensional search. However, for general redundant manipulators, the problem of conservativity and its overall effects on global optimality remains unsolved.\(^{18}\) Therefore, the proposed method is really only applicable to the case \(n - m = 1\). Under the assumption of the redundancy of one, let us denote the projection operator \((I - J^+J)\) in (15) as \(P\); then \(P\) can be described by the null basis vector \(N\) such that

\[
P = N(NN^T)^{-1}N^T
\]  

(16)

Using (16), (15) becomes

\[
\dot{\theta} = J^+ \dot{x} - \alpha N(NN^T)^{-1}N^T \nabla H
\]  

(17)

Since the second term of the right side in (17) has the rank of one, it can be parameterized by a scalar value \(\mu\) as

\[
\alpha N(N^TN)^{-1}N^T \nabla H = \mu N
\]  

(18)

where \(\mu = \alpha (N^TN)^{-1}N^T \nabla H\). Using (18), (17) becomes

\[
\dot{\theta} = J^+ \dot{x} - \mu N
\]  

(19)

In (19), \(N\) is any vector that spans the null space of \(J\). Indeed, to efficiently parameterize \(\dot{\theta}(t_0)\) we have to specifically define \(N\), so we choose \(N\) as the orthonormal vector obtained by the theorem of singular value decomposition.\(^{21}\)
Therefore, the optimization problem is to find an optimal parameter $\mu$ that yields the initial joint velocity minimizing the performance index while satisfying the periodic boundary conditions. If we find the parameter $\mu$ satisfying the periodic boundary conditions and minimizing the performance index, we can obtain the optimal trajectories over the entire task for the imposed initial configuration $\theta(t_0)$. In this case, if $\theta(t_0) = \theta(t_1)$ is satisfied then $\dot{\theta}(t_0) = \dot{\theta}(t_1)$ is preserved as long as $\dot{\theta}(t)$ is governed by (10).

In this article, we use (19) iteratively at the initial time to obtain an optimal initial joint velocity that satisfies the periodic boundary conditions. We define $H$ as the initial configuration constraint to be minimized for satisfying the periodic boundary conditions

$$H = \frac{1}{2} \| \theta(t_1) - \theta(t_0) \|^2$$

(20)

where $\theta(t_0)$ is the initial configuration and $\theta(t_1)$ is the final joint value evaluated at $t = t_1$ after forward integration from $t = t_0$ to $t = t_1$ of (10). Our optimal problem is to find the initial joint velocity $\dot{\theta}(t_0)$ that minimizes (20) while satisfying the kinematic constraints, $\dot{x}(t_0) = J\dot{\theta}(t_0)$, for the imposed initial configuration $\theta(t_0)$. Then the solution fulfills the periodic boundary conditions.

To fulfill the periodic boundary conditions, we denote $e$ as the configuration error between the initial and final time such that

$$e = \theta(t_1) - \theta(t_0)$$

(21)

where $e$ is the gradient of $H$ with respect to the final joint configuration. The key in our numerical search method is successive adjustment of the parameter $\mu$, which parameterizes the initial joint velocity, until the initial configuration constraint $H$ is minimized within a specified tolerance. On every iteration, the initial joint velocity may be updated by the configuration error. Since the degree of redundancy is one, the initial joint velocity is represented as $\dot{\theta}(t_0) = J^*\dot{x}(t_0) - \mu_0N$, where $\mu_0$ is the initial value of $\mu$, and the initial joint velocity on the $k$th iteration is updated as follows.

$$\dot{\theta}_k(t_0) = J^*\dot{x}(t_0) - \mu_kN$$

(22)

$$\mu_{k+1} = \mu_k + \sigma N^T e_k \quad \text{for} \ k = 0, 1, \ldots$$

(23)

where $0 < \sigma < 1$ is a positive real number and $e_k$ is the configuration error defined as (21) on the $k$th iteration.

In the iterative optimization procedure, the updating procedure is composed of two subprocedures: the calculation of a searching direction and the calculation of a step length. We note that $N^T e_k$ is the gradient of $H$ projected into the null space orthogonal to the rows of the Jacobian matrix, so this type of optimization is sometimes called the reduced-gradient type method. Since (22) with (23) satisfies the active constraints such as $\dot{x}(t_0) = J\dot{\theta}(t_0)$, the initial joint veloc-
ity during the searching process retains the differential kinematic constraints. Once \( \theta_k(t_0) \) is obtained, \( H(\mu_k) \leq H(\mu_{k-1}) \) must be verified so that \( \dot{\theta}_k(t_0) \) is indeed an improvement on \( H \). If the improvement is not made, damping should be introduced on the scaling step \( \Delta \mu_{k+1} = \mu_{k+1} - \mu_k = \sigma N^T e_k \) by a positive real number \( \sigma \) smaller than unity.

Because the iterative optimization algorithm is based on the gradient projection method, the following condition is achieved.

\[
\lim_{k \to \infty} N^T e_k = 0
\]  

(24)

From (24), the convergence criterion is \( ||e_k|| < \varepsilon \), where \( \varepsilon > 0 \) is a prescribed tolerance. By applying (22) and (23), we can determine the parameterized initial joint velocity \( \dot{\theta}_f(t_0) = J^* \dot{x}(t_0) - \mu_f N \), where \( \mu_f \) is the final value of the parameter. It should be noted that, although the numerical search provides an optimal parameter \( \mu_f \) with \( e = 0 \), a solution may arrive at a local point that satisfies (24). This situation occurs when \( e \) is orthogonal to the basis vector of the null space, i.e., \( e \) is not a zero vector. In this case, \( \mu_k \) in (23) is not updated, the initial joint velocity in (22) stays at a local minimum point, and we cannot obtain the initial velocity that satisfies the periodic boundary conditions. However, this situation can be easily avoided by readjusting the initial parameter \( \mu_0 \). It is interesting to note here that we can easily obtain an optimal solution for a sufficiently small task compared with the geometry of the manipulator. Therefore, the choice of \( \mu_0 \) that results in the \( \mu_f \) with \( e = 0 \) is easily obtained. This will be discussed through numerical examples in the next section.

4. NUMERICAL EXAMPLES

In this section, we discuss some features of the optimal redundancy resolution through numerical examples for a three-link planar manipulator. Consider the three-link planar manipulator in a horizontal plane shown in Figure 1. We choose the position of the end-effector in 2D space described in Cartesian coordinates; accordingly, \( x \in \mathbb{R}^2 \). The degree of redundancy at nonsingular points is equal to one. The manipulator has link lengths, \( l_1 = 3, l_2 = 2.5, \) and \( l_3 = 2 \) units, and masses \( m_1 = m_2 = m_3 = 1.0 \) kg. If we denote \( s_1 = \sin(\theta_1), c_1 = \cos(\theta_1), s_{12} = \sin(\theta_1 + \theta_2), \) and \( c_{12} = \cos(\theta_1 + \theta_2) \), the kinematic equations are

\[
\begin{bmatrix}
l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\
l_1 s_1 + l_2 s_{12} + l_3 s_{123}
\end{bmatrix}
\]  

(25)

For the three-link planar manipulator, the differential relation between the joint velocity and the end-effector velocity is obtained by differentiating (25) so that the Jacobian matrix \( J \) comprises partial derivatives of the function \( x \) with respect to joint values \( \theta_1, \theta_2, \) and \( \theta_3 \).

In the numerical examples, the cyclic tasks are described as
\[ \mathbf{x} = \begin{bmatrix} -R \cos(2\pi t) + C \\ -R \sin(2\pi t) \end{bmatrix} \]  

(26)

where \( R \) is the radius of a circle to be carried out and \( C \) is the \( x \)-axis position of the center of the circle. The task is to rotate the circle of \( R \) unit radius, centered at \((C, 0)\), in unit time counterclockwise; thus, the initial position is \((C - R, 0)\), and \( t_0 = 0 \) and \( t_1 = 1 \).

To compare with other results\(^{15,18}\) and demonstrate the effectiveness of the proposed method, we set \( M(\theta) = I \) and \( g(\theta) = 0 \). In this case, \( p(\theta, \dot{\theta}, t) = 1/2 \dot{\theta}^T \dot{\theta} \), and then the performance index (1) becomes

\[ r = \frac{1}{2} \int_{t_0}^{t_1} \dot{\theta}^T \dot{\theta} \, dt \]  

(27)

The necessary condition for (27) can be easily obtained from (10) such that

\[ \ddot{\theta} = J^+(\ddot{x} - J\dot{\theta}) \]  

(28)

where \( \ddot{\theta} \) is the solution that minimizes the joint velocity norm along the path.

### 4.1. Nonhomotopic Extremal Solutions

Consider a task shown in Figure 1 where \( R = 1 \) and \( C = 6 \) units. In this task denoted as Task 1, we have the initial configuration \( \theta(t_0) = (0.7854, -0.8488, -1.3143)^T \) radians. In this initial configuration, the null space basis \( \mathbf{N} = (0.317, -0.644, 0.696)^T \) is obtained by the singular value decomposition of \( J \). First, to illustrate the initial value adjustment problem, we consider \( H \) for the imposed initial configuration and the computed initial joint velocity using (19) with arbi-
Figure 2. $H$ (solid line) and $r$ (broken line) for Task 1.

To reduce the range of $\mu$, we may use the manipulability measure. Since this measure indicates the manipulating ability of a manipulator in positioning and orienting the end-effector, we consider $H$ for ranged $\mu$ according to the manipulability measure. The manipulability measure denoted as $M_m$ for the above initial configuration is 15.24.

Figure 2 shows $r$, which is scaled down by 0.02, and $H$ with respect to ranged $\mu$, i.e., $-M_m \leq \mu \leq M_m$. As shown in Figure 2, we may find multiple extremal solutions around $\mu$s such as $0$, $5$, $10.5$, $-5$, $-10.5$, and so on, at which periodic boundary conditions are satisfied, but only one of them is the optimal solution. To obtain a conservative optimal solution, $H$ should reside in a preassigned tolerance, and $r$ must be minimized. So, it is expected that the globally optimal trajectory can be obtained around $\mu = 0$, which corresponds to the minimum norm solutin of the Jacobian relation.

In Figure 2, several local optima may exist and the corresponding performance values may differ substantially. The problem of designing an algorithm that distinguishes between local optima and locates the best possible one is known as the global optimization problem. The global optimization method, based on generating a sequence of local minima with decreasing function values such as the tunneling method, may be applicable to this problem. In spite of the fact that the global minimum can be obviously found among local ones, global optimization is difficult because the performance index is an integral-type criterion and the initial configuration constraint due to conservativity requirement should be evaluated after integration of (28).

Therefore, we give a general approach for determining the initial parameter $\mu_0$. As discussed in the previous section, the natural boundary conditions such
as (14) yield the minimum norm solution of the Jacobian relation. The solution (28) subject to the natural boundary conditions gives the minimum value of the performance index $r$. Since the minimum norm solution is equivalent to the joint velocity with $\mu = 0$ in the proposed method, the iterative optimization algorithm of (22) and (23) starting from $\mu_0 = 0$ gives the optimal value $\mu_f$ that parameterizes the initial joint velocity satisfying the periodic boundary conditions. Therefore, our proposed method is modified from the problem of the natural boundary conditions to the initial configuration-constrained problem for providing conservative joint motions.

Based on the above discussion, we may determine an optimal parameter by using (22) and (23) around $\mu_0 = 0$ with $\sigma = 0.5$, which results in $\mu_f = -0.0345$ at which we can obtain an optimal solution. If the cost of the joint trajectory generated by this $\hat{\theta}(t_0)$ for the imposed $\theta(t_0)$ is lower than those of all other locally optimal $\hat{\theta}(t_0)$, then it is a globally optimal solution. Also, if the search for the initial joint velocity starts from $\mu_0 = 5$, then a locally optimal solution, which is $\mu_f = 4.88272$, will be obtained with the same initial configuration. Figure 3 shows two different solutions for the imposed initial configuration: One is optimal, denoted as Solution A, when $\mu_f = -0.0345$ and the other is extremal, denoted as Solution B, when $\mu_f = 4.88272$. As shown in Figure 3, the globally optimal solution is smoother than the locally optimal solution.

To better understand the behaviors of solutions in the joint space, we depict the set of joint values on the $\theta_3 - \theta_2$ plane. Figure 4 shows joint trajectories on the $\theta_3 - \theta_2$ plane to compare an optimal solution with an extremal solution. From the geometry of the manipulator shown in Figure 1, we can easily find singular points, at which the three links are colinear. Burdick\textsuperscript{23} showed that inverse kinematic solutions of a redundant manipulator at a nonsingular point $x$

![Figure 3. Two different solutions for Task 1, $\theta(t_0) = (0.7854, -0.8488, -1.3143)^T$.](image)
must be an $n - m$ dimensional submanifold of the configuration space. Each of the disjoint manifolds is said to be self-motion manifold (sometimes called the set of the homogeneous solutions). Since Task 1 is described as a circle between inner singular position $x = 3.5$ and the boundary of work volume, the set of inverse kinematic solutions according to $x(t)$ forms one self-motion manifold. Therefore, the outer closed curve and inner closed curve are two sets of the inverse kinematic solutions at $t = t_0$ and $t = t_1/2$, respectively. For cyclic tasks of (26), any joint trajectory should start from the outer closed curve, hit the inner closed curve, and then return to the starting point again.

Based on Figure 4, the globally optimal solution is shorter than other trajectories for the same initial configuration in the joint space. For Task 1, Solution A resides only in third quadrants, and arm posture does not change through the task. However, Solution B encloses the inner closed curve through all quadrants; therefore, arm posture varies. It is important to note that the longer trajectory cannot be transformed continuously into the shorter trajectory because the inner closed curve forms an $I$-manifold in the configuration space; therefore, two solutions are nonhomotopic. The notion of homotopic has previously been used to characterize different redundancy resolution paths. In the language of differential topology, the notion of homotopic is as follows. If $f_0$ and $f_1$ are continuous maps of the space $X$ into the space $Y$, we say $f_0$ is homotopic to $f_1$ if there is a continuous map $F: X \times I \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for each $x \in X$ (here $I = [0, 1]$). Also, if two paths are homotopic and have the same initial and final points, then it is called path homotopic. Therefore, the solutions in Figure 4 are path nonhomotopic. And, the paths that are homotopic to each other form a homotopy class. The homotopy classes are,
therefore, classified by a parameter \( \mu \) that parameterizes the initial joint velocity as shown in Figure 2. In this example, we can find the globally optimal solution in the desirable homotopy class, which is about \(-3 \leq \mu \leq 3\).

If we want to carry out Task 1 at another initial configuration \( \theta(t_0) = (-0.47124, 1.7875, -1.8734)^T \) radians, we should determine initial joint velocity for satisfying periodic boundary conditions. In this configuration, \( N = (0.4843, -0.5366, -0.6910)^T \) and \( M_m = 9.85 \). It should be noted that the initial value of \( \mu \), which converges to \( \mu_f \) giving an optimal solution, does not depend on the initial configuration but on the task. Roughly speaking, the initial joint velocities satisfying the conservativity requirement for the same task are similar in spite of different initial configurations so that the same values of \( \mu_0 \) and \( \sigma \) of Solution A can be used to obtain an optimal solution for different initial configurations. Then, we can find the globally optimal solution at \( \mu_f = -0.004345 \), which is shown as Solution C in Figure 5a. If we use a value much larger than zero for \( \mu_0 \), we can obtain another extremal solution for this initial configuration. To compare optimal solutions for different initial configurations, i.e., Solution A and Solution C, Figure 5b shows Solution C on the \( \theta_3 - \theta_2 \) plane.

So far, we have shown the way of obtaining the globally optimal solution among various extremal solutions using the parameterized initial joint velocity. And, we have found that the solutions of the optimal redundancy resolution problem are path nonhomotopic for different initial joint velocities, as well as nonhomotopic for different initial joint configurations.

4.2. Assignment of the Initial Parameter

To investigate the convergence of the proposed method for an arbitrary task, we consider a cyclic task of \( R = 1.0 \) and \( C = 1.2 \) units as Task 2 whose location is different from Task 1. In this example, the initial joint value is \( \theta(t_0) = (1.300, -2.343, -1.725)^T \) radians and the manipulability measure is \( M_m = 4.98 \). The null space basis is \( N = (0.9928, -0.0294, 0.1162)^T \). As discussed in the previous examples, we may search a parameter from \( \mu_0 = 0 \) to produce an optimal parameter. If an optimal parameter is determined by searching from \( \mu_0 = 0 \) with \( \sigma = 0.4 \), then it results in \( \mu_f = 1.6297 \). It is interesting to note that \( \mu_f \) obtained for Task 2 is larger than those for Task 1 because \( M_m \) for Task 2 is smaller than those for Task 1. Thus, the optimal solution of Task 2 for this initial configuration, denoted as Solution D in Figure 6, has the large initial joint velocity in comparison with the optimal solutions for Task 1. This means that the initial joint velocity becomes large as the end effector nears the base of the manipulator at the beginning and end of the motion.

Consider another cyclic task of \( R = 2.3 \) and \( C = 2.5 \) units denoted as Task 3. It has the same initial position as Task 2, but its radius is larger than that of Task 2. If we start searching from \( \mu_0 = 0 \) with \( \sigma = 0.1 \), the parameter \( \mu \) does not converge to \( \mu_f \) with \( \epsilon = 0 \). This situation is discussed in Section 3 through the relationship between the configuration error and the null space basis of the Jacobian matrix. In a practical sense, since the initial joint velocity must be
larger than that of Task 2, the minimum norm joint velocity is insufficient to carry out Task 3 while satisfying the periodic boundary conditions.

To overcome the orthogonality between the null space basis $N$ and the configuration error $e$, we suggest an efficient algorithm for determining $\mu_\Phi$. Using homogeneous solutions in the velocity vector of (19) involves a trade-off between the minimization of the joint velocity and optimization of an objective function. Even though the initial position of the end effector is the same, we
should consider the end-effector velocity according to the changed task. If one has $\mu_f$ that produces the optimal solution in the desirable homotopy class, it can be used to obtain the optimal solution for the changed task as far as the initial position of the end effector is the same. In this case, we may use the manipulability-velocity ratio (MVR), which is the ratio of the norm of the end-effector velocity to the norm of joint velocity.\textsuperscript{6} If $r_v$ is referred to as MVR, then it is defined as

$$
r_v^2 = \frac{\ddot{x}^T \dot{x}}{\dot{\theta}^T \dot{\theta}}
$$

Using the initial joint velocity represented as (19), $r_v$ at the initial time becomes

$$
r_v^2 = \frac{\ddot{x}^T \dot{x}}{\dot{x}^T (J^T)J^{-1} \dot{x} + \mu^2}
$$

For a given $r_v$, we take a positive value $\mu$ for obtaining the conservative optimal solution because we want to minimize the initial configuration constraint $H$ using (19).

To show how to determine the initial parameter $\mu_0$ for Task 3, we use $\mu_f$ obtained as for Task 2. Since $\mu_f = 1.6297$ for Task 2, $r_v^2 = 2.346$ can be obtained according to $\dot{x}(t_0)$ of Task 2. The manipulator requires about the above $r_v$ to produce conservative joint motions for the given initial end-effector velocity and the initial configuration. To preserve the manipulability-velocity ratio, we should start searching from $\mu_0 = 3.75$ for Task 3, which resulted from (30) using
of Task 2. By using the iterative algorithm (22) and (23), we can find \( \mu_f = 5.62 \) with which eq. (28) gives an optimal solution. To better understand the feature of the optimal solution for a large task, the joint trajectory denoted as Solution E is also shown in Figure 6. As shown in Figure 6, the joint trajectory for the large task yields a conservative joint motion and preserves arm posture during the task.

4.3. Topological Lifting of the Paths

As mentioned before, the true optimal trajectory among all feasible trajectories satisfying the periodic boundary conditions may be difficult to find because there exist multiple nonhomotopic extremal solutions even for the imposed initial configuration. Therefore, we summarize an algorithm to obtain a globally optimal solution in the desirable homotopy class. The algorithm consists of two steps: Step 1 is to obtain an optimal solution for an arbitrary task with an imposed initial configuration and step 2 is to get an optimal solution of another initial configuration for the same task by using the parameter obtained in step 1.

Step 1. Since the initial joint velocity depends on the task, \( \mu_0 \) must be chosen according to the task. For small tasks such as Tasks 1 and 2, one may find \( \mu_f \), which produces an optimal trajectory, around minimum norm joint velocity, i.e., \( \mu_0 = 0 \). Therefore, for small tasks we do not need to find configuration errors at the final time for arbitrary \( \mu \) such as Figure 2. If the circle to be traced has large radius, for example, we may use the MVR in (30) to determine new \( \mu_0 \) for a changed task. In this way, we can easily obtain the optimal trajectory for the imposed initial configuration.

Step 2. If the parameter \( \mu_f \) that produces an optimal trajectory is found from \( \mu_0 \) by Step 1, \( \mu_0 \) can be used to find other \( \mu_f \) that give an optimal trajectory for different initial configurations of the same task while excluding extremal solutions. This procedure is a topological lifting of the paths\(^{15}\) for the same task so that we can obtain the optimal solution by searching from \( \mu_0 \), which is in the desirable homotopy class.

As shown previous examples, Solution C in Figure 5(b) has been found by using the same parameter \( \mu_0 \) obtained as Solution A in Figure 4.

4.4. Consideration of the Dynamics

So far, we have discussed the optimal redundancy resolution considering only the kinematics of a manipulator. However, if a trajectory is required to consider the dynamic response, the dynamics of a manipulator must be taken into consideration. For optimal resolution, we may consider a kinetic energy minimization problem, such as \( p(\theta, \dot{\theta}, t) = \frac{1}{2} \dot{\theta}^T M(\theta) \dot{\theta} \), subject to the kinematic constraints, where \( M(\theta) \) is the inertial matrix of the dynamic equation.
For this performance index $p$, the necessary condition becomes (10) with $p = 0$. The inertial matrix $M(\theta)$ when the links are modeled by a point mass at the distal end of each link and $V_{cc}$ are described in the Appendix.

By the numerical search method discussed in Section 3, we can obtain the initial joint velocity that uniquely defines the joint trajectory for the imposed initial configuration. For comparison, consider the case of Solution A for Task 1. The optimal parameter is searched around the minimum norm joint velocity, which is at $\mu_0 = 0$; then we obtain a globally optimal solution at $\mu_f = 0.00971$. In this case, the performance index given by the kinetic energy (computed in a discretized form) is 35.088 and that of Solution A is 36.824. To better understand the dynamic response, we depict the motions of the manipulator in Figure 7. We note that the motion of joint one in Figure 7b is smaller than that of Solution A in Figure 7a so that the kinetic energy is minimized by considering the dynamics of the manipulator.

**Figure 7.** (a) Motions of Solution A for Task 1 when only kinematics is considered. (b) Motions of solution for Task 1 when dynamics is taken into account.
5. CONCLUSIONS

In this article, the optimal redundancy resolution of a redundant manipulator for providing conservative joint motions has been worked out by using a general integral-type criterion. The necessary conditions using the calculus of variations result in a second-order differential equation. For a cyclic task, we considered the periodic boundary conditions due to the conservativity requirement and then refined the periodic boundary problem to the initial value adjustment problem, at which point we should find the initial joint velocity for the imposed initial configuration.

To satisfy the periodic boundary conditions, as well as the necessary conditions, we presented a numerical search method for obtaining a globally optimal conservative trajectory. And, the characteristics of the iterative optimization algorithm have been discussed. We parametrized the initial joint velocity to reduce the search dimension. The initial value adjustment problem is to determine an optimal parameter that minimizes the initial configuration constraint. Based on the relationships between the natural boundary conditions and the proposed method, we discuss a general approach for choosing the initial parameter of the joint velocity. We note that the proposed method is only directly applicable to the case $n - m = 1$.

Through numerical examples, the existence of nonhomotopic extremal solutions according to initial joint velocities for the same initial configuration was discussed. For changed tasks, we could find the globally optimal solution using the manipulability-velocity ratio as long as the initial position of the end effector is the same. And, we depicted an optimal trajectory by considering the dynamics of a three-link planar redundant manipulator, which produces the improved dynamic response. Therefore, the proposed method, which is refined to an initial value adjustment problem with the initial configuration constraint, can be applied to the general performance criteria such as the least-squares joint velocities and the least kinetic energy.

APPENDIX

For the three-link planar manipulator, the inertial matrix $M(\theta) = [m_{ij}] \in \mathbb{R}^{3 \times 3}$ and Coriolis and centrifugal torque vector $\mathbf{V}_{cc} = [v_i] \in \mathbb{R}^3$ can be described as follows.

$$m_{31} = m_3 l_3 (l_1 c_{23} + l_2 c_3 + l_3)$$

$$m_{32} = m_3 l_3 (l_2 c_3 + l_3)$$

$$m_{33} = m_3 l_3^2$$

$$m_{21} = m_{31} + m_2 l_2 (l_1 c_2 + l_2) + m_3 l_2 s_3 (l_1 s_{23} + l_2 s_3) + m_3 l_2 c_3 (l_1 c_{23} + l_2 c_3 + l_3)$$

$$m_{22} = m_2 l_2^2 + m_3 (2l_2 l_3 c_3 + l_3^2 + l_2^2)$$
\[ m_{23} = m_3(l_5^3 + l_2l_3c_3) \]
\[ m_{11} = m_1l_1^3 + m_2(l_1l_2c_2 + l_3^2 + l_5^2) + m_3(l_1^2 + l_2^2 + l_3^2 + 2l_1l_2c_2 + 2l_1l_3c_{23} + 2l_2l_3c_3) \]
\[ m_{12} = m_2(l_3^2 + l_1l_2c_2) + m_3(l_2^2 + l_3^2 + 2l_2l_3c_3 + l_1l_2c_2 + l_1l_3c_{23}) \]
\[ m_{13} = m_3(l_2^3 + l_2l_3c_3 + l_1l_3c_{23}) \]

Let us denote \((\dot{\Theta} \dot{\Theta}) = (\dot{\theta}_1 \dot{\theta}_2 \dot{\theta}_3 \dot{\theta}_4 \dot{\theta}_5) = [\dot{\Theta}] \in \mathbb{R}^3\); then \(v_i\) can be represented as

\[ v_i = \sum_{j=1}^{3} (b_{ij}\dot{\theta}_j^2 + c_{ij}\dot{\theta}_j) \]

where \(b_{ij}\) and \(c_{ij}\) are elements of centrifugal matrix and Coriolis matrix, respectively, which are described as

\[ b_{31} = m_3l_3(l_1s_{23} + l_2s_3) \]
\[ b_{32} = m_3l_2l_3s_3 \]
\[ b_{33} = 0 \]
\[ b_{21} = b_{31} + l_1l_2m_2s_2 - m_3l_2s_3(l_1c_{23} + l_2c_3 + l_3) + m_3l_2c_3(l_1s_{23} + l_2s_3) \]
\[ b_{22} = b_{32} + l_2m_3[l_2c_3s_3 - s_3(l_2c_3 + l_3)] \]
\[ b_{23} = -m_3l_2l_3s_3 \]
\[ b_{11} = b_{21} - m_3l_1s_{23}(l_1c_{23} + l_2c_3 + l_3) + m_3l_1c_{23}(l_1s_{23} + l_2s_3) + m_2l_1[l_1c_2s_2 - s_2(l_1c_2 + l_2)] \]
\[ b_{12} = b_{22} - m_3l_1s_{23}(l_2c_3 + l_3) + m_3l_1l_2s_3c_{23} - m_2l_1l_2s_2 \]
\[ b_{13} = b_{23} - m_3l_1l_3s_{23} \]
\[ c_{31} = 2m_3l_3s_3 \]
\[ c_{32} = 0 \]
\[ c_{33} = 0 \]
\[ c_{21} = c_{31} - 2m_3l_3s_3(l_2c_3 + l_3) + 2m_3l_2^2c_3s_3 \]
\[
c_{22} = -2m_3l_2s_3 \\
c_{23} = c_{22} \\
c_{11} = c_{21} + 2l_1[m_3(l_2c_{23}s_3 - s_{23}(l_2c_3 + l_3)) - m_2l_2s_3] \\
c_{12} = c_{22} - 2m_3l_1s_3s_3 \\
c_{13} = c_{12}
\]

References


